

Factorization in the model of unstable particles with smeared masses

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We study the processes with unstable particles in intermediate states. It is shown that the amplitudes squared of such processes factor exactly in the framework of the model of unstable particles with smeared masses. Decay widths and cross sections can then be represented in a universal factorized form for an arbitrary set of interacting particles. This exact factorization is caused by specific structure of propagators in the model. We formulate the factorization method and perform a phenomenological analysis of the factorization effects. The factorization method considerably simplifies calculations while leading to compact and reasonable results.

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I. INTRODUCTION

Description of unstable particles (UP's) with a large width runs into some problems which have been under considerable discussion for many decades (see [1–10] and references therein). These problems are both conceptual and computational in character and arise due to the fact that UP's lie somewhat outside the traditional formulation of quantum theory [3, 6, 10]. An unstable particle with a large width cannot be treated as an asymptotic initial or final state. So, UP is usually described in an intermediate state with the help of the dressed propagator or the S -matrix with a complex pole. However, the application of the Dyson procedure leads to a deviation from the scheme of fixed-order calculations and to some problems with gauge cancellation [11–13]. In order to overcome these obstacles, some approximation schemes and methods have been worked out. For instance, double-pole approximation (DPA) [14–17], pinch-technique method [18, 19], complex-mass scheme (CMS) [20, 21], semi-analytical approximation [13], narrow-width approximation (NWA) [13, 22], convolution method (CM) [23–28], etc. At the same time, alternative ways to deal with UPs were developed, such as effective theories of UP's [29, 30], modified perturbation theory [9] and the model of UP's with a smeared mass [31].

The properties of UP have also been a subject of discussion in the literature. In particular, the assumption that the decay of unstable particle R proceeds independently of its production remains to be of interest [32–34]. Formally, this effect is expressed quantitatively as the factorization of a cross section or decay width [34]. The processes of the type $ab \rightarrow Rx \rightarrow cdx$ were considered in Ref. [34]. It was shown that in the framework of the traditional approach the factorization is always valid for a scalar R and does not take place for a vector or spinor R . The effect of the factorization is usually considered as related to NWA [13], which is based on five assumptions [22].

In this work, we analyze the factorization effects in the framework of the model of Ref. [31]. In the previous papers [35, 36], it was shown by straightforward calculations that the expressions for the width of the decay $a \rightarrow bR \rightarrow bcd$ and for the cross section of the process $ab \rightarrow R \rightarrow cd$ factor exactly. The exact factorization is caused by the specific structure of the propagators of UP's with a smeared mass. The factorization method based on these results was developed in [37, 38]. Note that the discussion in Refs. [35–38] was limited to some simplest tree diagrams. Here, we consider the general case—vertexes have arbitrary (loop) structure and UP carries an arbitrary spin j_R . It is shown that the amplitude squared can be exactly factorized upon integration over the phase space of the final states. On the basis of this analysis, the factorization method [38] receives further development.

The paper is organized as follows. In Sec. II, we systematically study the factorization of the amplitudes squared for the processes with UP in an intermediate time-like state. Universal factorized formulae for decay widths and cross sections are derived in Sec. III. The factorization approach is applied to the processes with several consequent decays of UP's in Sec. IV, where the factorization method is formulated in its most general form. In Sec. IV, we also discuss some methodological and phenomenological aspects of the factorization. Appendices give details of the mathematical proofs.

II. FACTORIZATION EFFECTS IN THE MODEL OF UNSTABLE PARTICLES WITH SMEARED MASSES

In this section, we consider the structure of the amplitudes for the processes $X_I \rightarrow R \rightarrow X_F$ (X_I and X_F are arbitrary sets of initial and final states; R stands for an unstable particle in an intermediate state transferring a time-like momentum). We show that a special form of the propagators of unstable fields leads to the factorization of transition probabilities. In contrast to the traditional treatment (the narrow-width approximation), the approach under consideration provides the exact factor-

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ization for any type of UP. This effect makes it possible to represent decay widths and cross sections in a universal factorized form.

In the model considered, the propagators of scalar, spinor, vector, and vector-spinor unstable fields are defined by the following expressions [31, 38]:

$$\begin{aligned} D(q) &= \frac{i}{P(q)}, & \hat{D}(q) &= i \frac{\not{q} + q}{P(q)}, \\ D_{\mu\nu}(q) &= -i \frac{g_{\mu\nu} - q_\mu q_\nu / q^2}{P(q)}, \\ \hat{D}_{\mu\nu}(q) &= -i \frac{\not{q} + q}{P(q)} \left[g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu \right. \\ &\quad \left. - \frac{\gamma_\mu q_\nu - \gamma_\nu q_\mu}{3q} - \frac{2}{3} \frac{q_\mu q_\nu}{q^2} \right], \end{aligned} \quad (1)$$

where $q = \sqrt{q_\mu q^\mu}$.

In the general case, the propagators of the fields carrying integer spins $j = \ell$ and half-integer ones $j = \ell + 1/2$ ($\ell = 1, 2, \dots$) can be written, respectively, as

$$\begin{aligned} D_{\bar{\mu}\bar{\nu}}(q) &= -\frac{i}{P(q)} P_{\bar{\mu}\bar{\nu}}^{(\ell)}(q), \\ \hat{D}_{\bar{\mu}\bar{\nu}}(q) &= -i \frac{\not{q} + q}{P(q)} P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q). \end{aligned} \quad (2)$$

Here, $\bar{\mu} = \mu_1 \mu_2 \dots \mu_\ell$, $\bar{\nu} = \nu_1 \nu_2 \dots \nu_\ell$ are multi-indices; $P_{\bar{\mu}\bar{\nu}}^{(j)}$ are the spin- j projection operators defined in Appendix A 3.

In Eqs. (1) and (2) the denominators $P(q)$ can be taken in various forms (pole, Breit–Wigner and other definitions). It is essential that the effect of factorization does not depend on the form of the denominators $P(q)$, but depends crucially on the tensor-spinor structure of the numerators in Eqs. (1) and (2). The phenomenological propagators with a smeared mass parameter q result in the exact factorization, while conventional field-theory expressions with a constant mass M lead to an approximate factorization in the narrow-width limit. It should be noted that the structures (1) and (2) are not constrained by the choice of the gauge. The model under consideration is not a gauge one, because it describes effective unstable fields (see Ref. [31] for details). We also note that the difference between the model and field-theory functions is given by the value $(q^2 - M^2)/M^2$, which is small in the vicinity of a resonance. Moreover, it is strongly suppressed by additional small factors [38]. So, the approach discussed can be treated as some close approximation to the standard one, that is, it gives an analytical alternative to NWA (see Sec. IV and [38]).

The factorization can be illustrated diagrammatically by cutting an internal line that stands for UP with a time-like momentum, if such an operation disconnects the diagram into two independent fragments. So, in the framework of the phenomenological model under consideration we need to include UP in initial and final states. The polarization density matrices of vector, spinor, and

vector-spinor UPs are defined as the following spin sums [37, 38]:

$$\begin{aligned} \Pi_{\mu\nu}(q) &= \sum_{a=1}^3 e_\mu^a(\mathbf{q}) \bar{e}_\nu^a(\mathbf{q}) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}, \\ \hat{\Pi}(q) &= \sum_{a=1}^2 u^{a,\mp}(\mathbf{q}) \bar{u}^{a,\pm}(\mathbf{q}) = \frac{\not{q} \pm q}{2q^0}, \\ \hat{\Pi}_{\mu\nu}(q) &= -\frac{\not{q} \pm q}{2q^0} \left[g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu \right. \\ &\quad \left. - \frac{\gamma_\mu q_\nu - \gamma_\nu q_\mu}{3q} - \frac{2}{3} \frac{q_\mu q_\nu}{q^2} \right]. \end{aligned} \quad (3)$$

For higher spins $j \geq 1$ the operators (3) become

$$\begin{aligned} \Pi_{\bar{\mu}\bar{\nu}}(q) &= -P_{\bar{\mu}\bar{\nu}}^{(\ell)}(q), \\ \hat{\Pi}_{\bar{\mu}\bar{\nu}}(q) &= -\frac{\not{q} \pm q}{2q^0} P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q). \end{aligned} \quad (4)$$

Further, we show that the similarity of the propagators (1), (2) and the polarization matrices (3), (4) leads to the exact factorization of the cross sections and decay widths in a few simplest special cases. In Appendix A, we generalize the result to any interaction mediated by UP with arbitrarily high spin.

To give an example of the factorization, let us consider first the simplest two-particle scattering $ab \rightarrow R \rightarrow cd$, where R is UP with a large width (see Fig. 1). In the case of a spinor UP we can prove by simple calculation that the amplitude squared of the process $\phi_1 \psi_1 \rightarrow \psi_R \rightarrow \phi_2 \psi_2$ (ϕ_a and ψ_a are scalar and fermion particles, respectively) can be written in a factorized form

$$|\mathcal{M}|^2 = \frac{4(q^0)^2}{|P_R(q)|^2} |\mathcal{M}_1|^2 \cdot |\mathcal{M}_2|^2. \quad (5)$$

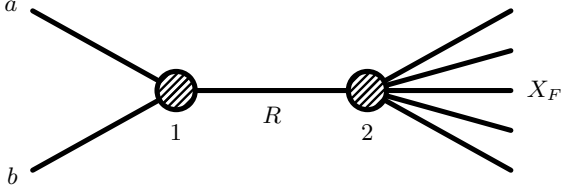
Here, \mathcal{M}_1 and \mathcal{M}_2 are the amplitudes of the processes $\psi_R \rightarrow \phi_1 \psi_1$ and $\psi_R \rightarrow \phi_2 \psi_2$, respectively. The amplitudes squared $|\mathcal{M}|^2$ and $|\mathcal{M}_i|^2$ contain traces of 4×4 spinor matrices and the factorization in Eq. (5) is caused by the factorization of the trace

$$\begin{aligned} &\text{Tr} \left[(\not{k}_1 + m_2)(\not{q} + q)(\not{p}_1 + m_1)(\not{q} + q) \right] \\ &= 2 \text{Tr} \left[(\not{k}_1 + m_2)(\not{q} + q) \right] \cdot \text{Tr} \left[(\not{p}_1 + m_1)(\not{q} + q) \right], \end{aligned} \quad (6)$$

where $q = p_1 + p_2$; p_1 and k_1 are the momenta of spinor particles in the initial and final states; $m_{1,2}$ are their masses.

By direct calculations it can be checked that the factorized relation (5) is not valid for the process $\phi_1 V_1 \rightarrow V_R \rightarrow \phi_2 V_2$, where ϕ denotes scalar fields and V are vector fields. However, the factorization is restored upon integrating over the phase space of the final states (ϕ_2, V_2):

$$J(|\mathcal{M}|^2) = \frac{4(q^0)^2}{|P_V(q)|^2} |\mathcal{M}_1|^2 \cdot J(|\mathcal{M}_2|^2). \quad (7)$$

FIG. 1. The scattering $ab \rightarrow R \rightarrow X_F$.

Here, the operator $J(A)$ denotes the integration over the phase space of the momenta k_1 and k_2 (see Eq. (A8) in Appendix A). The relation (7) can be verified by direct calculations, however, we prove it in a more general way. In the case under consideration, the amplitude squared $|\mathcal{M}(p, q, k)|^2$ is given by

$$\begin{aligned}
 |\mathcal{M}(p, q, k)|^2 &= \frac{12(q^0)^2}{|P_R(q)|^2} M_{(1)}^{\mu\nu}(p, q) M_{\mu\nu}^{(2)}(k, q), \\
 |\mathcal{M}_1(p, q)|^2 &\equiv M_{(1)\mu}^\mu(p, q) = \frac{g_1^2}{24q^0 p_a^0 p_b^0} \eta_{\mu\nu}(p_a) \eta^{\mu\nu}(q), \\
 |\mathcal{M}_2(k, q)|^2 &\equiv M_{(2)\mu}^\mu(k, q) = \frac{g_2^2}{24q^0 k_1^0 k_2^0} \eta_{\mu\nu}(k_1) \eta^{\mu\nu}(q).
 \end{aligned} \tag{8}$$

Here, g_k ($k = 1, 2$) are coupling constants; $\eta_{\mu\nu}(q) = -g_{\mu\nu} + q_\mu q_\nu / q^2$. Integrating $M_{\mu\nu}^{(2)}(k, q)$ over the final momenta k_1 and k_2 gives

$$\begin{aligned}
 J[M_{(2)}^{\mu\nu}(k, q)] &\equiv \iint d\mathbf{k}_1 d\mathbf{k}_2 \delta(q - k_1 - k_2) M_{(2)}^{\mu\nu}(k, q) \\
 &= c_1(q) g^{\mu\nu} + c_2(q) q^\mu q^\nu.
 \end{aligned} \tag{9}$$

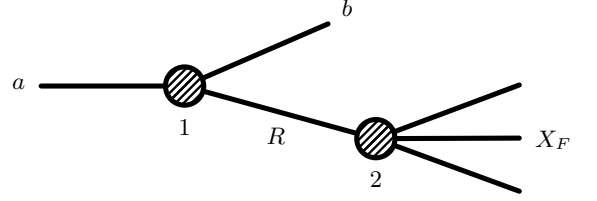
Multiplying Eq. (9) by $\eta^{\mu\nu}(q)$ we get

$$c_1(q) = \frac{1}{3} J[M_{(2)\mu}^\mu(k, q)] = \frac{1}{3} J[|\mathcal{M}_2(k, q)|^2]. \tag{10}$$

It is immediately seen that Eq. (7) follows from Eqs. (8), (9), and (10). Generalizing this result to the case of an arbitrary multi-particle final state and arbitrary structure of the vertices (which might include loop contributions) is considered in Appendix A. In this general case (see Appendix A), Eq. (7) becomes

$$\begin{aligned}
 J[|\mathcal{M}(p, k, q)|^2] &= \frac{(2j_R + 1)(2q^0)^2}{(2j_a + 1)(2j_b + 1)|P_R(q)|^2} \\
 &\times |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2].
 \end{aligned} \tag{11}$$

Here, $j_{a,b}$ are the spins of the initial particles, j_R is an integer spin of the resonance, and J denotes the integration over the multi-particle phase space. It should be noted that a stronger factorized relation is valid for a broad class of interactions of UPs carrying half-integer spins (including gauge and Fermi's interactions of fermions). In this case, the same factorization (11) of the amplitude squared exists even without the integration J over the phase space of the final particles.

FIG. 2. Factorization in $a \rightarrow bX_F$ decay diagram.

III. FACTORIZED FORMULAE FOR THE OBSERVABLES

For any kinds of particles in the initial $X_I = (a, b)$, final X_F , and intermediate R states the cross section of the scattering $ab \rightarrow R \rightarrow X_F$ (Fig. 1) can be written in a factorized form (see Appendix B):

$$\begin{aligned}
 \sigma(ab \rightarrow R \rightarrow X_F) &= \frac{16\pi L_R}{L_a L_b \bar{\lambda}^2(m_a^2, m_b^2; s)} \frac{1}{|P_R(s)|^2} \\
 &\times \Gamma(R(s) \rightarrow ab) \Gamma(R(s) \rightarrow X_F).
 \end{aligned} \tag{12}$$

Here, $L_i = 2j_i + 1$; $s = q^2$; $R(s)$ is UP with a smeared mass \sqrt{s} ; $\bar{\lambda}(m_a^2, m_b^2; s)$ is an analog of the Källén function

$$\bar{\lambda}(p_a^2, p_b^2; s) = \left[1 - 2 \frac{m_a^2 + m_b^2}{s} + \frac{(m_a^2 - m_b^2)^2}{s^2} \right]^{1/2}. \tag{13}$$

The cross section (12) does not depend explicitly on the spins of the final states. Therefore, it can be summed over the final channels $R(s) \rightarrow X_F$

$$\sigma_{\text{inclusive}}(ab \rightarrow R) = \frac{16\pi^2 L_R \Gamma(R(s) \rightarrow ab)}{L_a L_b \sqrt{s} \bar{\lambda}^2(m_a^2, m_b^2; s)} \rho_R(s), \tag{14}$$

where the function

$$\rho_R(s) = \frac{\sqrt{s} \Gamma_R^{\text{tot}}(s)}{\pi |P_R(s)|^2} \tag{15}$$

is interpreted in the model of UP with smeared mass as probability density of UP mass [31].

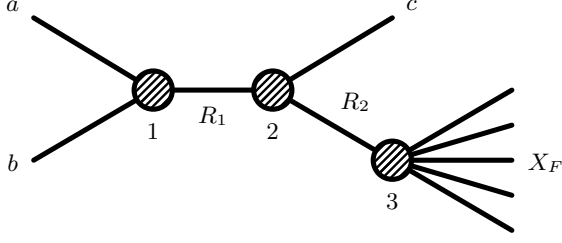
Now, we consider the decay process of type $a \rightarrow bR \rightarrow bX_F$, where R is UP with an arbitrary spin, a and b are quasi-stable particles with negligible width, and X_F is the set of final states (see Fig. 2).

For the decay process $a \rightarrow bR \rightarrow bX_F$ the relation similar to Eqs. (11) holds (see Appendix A):

$$J[|\mathcal{M}(p, k, q)|^2] = \frac{4(q^0)^2}{|P_R(q)|^2} |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2], \tag{16}$$

where $\mathcal{M}_1(p, q) = \mathcal{M}_1(a \rightarrow bR)$, $\mathcal{M}_2(k, q) = \mathcal{M}_2(R \rightarrow X_F)$. We then can calculate the width of the decay (see Appendix B):

$$\Gamma(a \rightarrow bX_F) = \int \Gamma(a \rightarrow bR(q)) \frac{q \Gamma(R(q) \rightarrow X_F)}{\pi |P_R(q)|^2} dq^2. \tag{17}$$

FIG. 3. Factorization in $ab \rightarrow cX_F$ scattering-decay diagram.

In the above equation, the domain of integration is constrained by the kinematics of the reaction. Summing over all the decay channels of UP we get

$$\Gamma(a \rightarrow bR) = \int \Gamma(a \rightarrow bR(q)) \rho_R(q^2) dq^2, \quad (18)$$

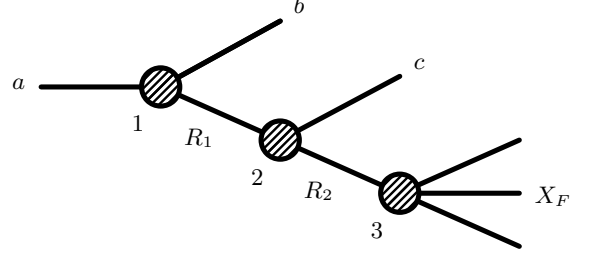
where the function $\rho_R(q^2)$ is defined above in Eq. (15). Note that the factorization (12) and (17) is exact within the framework of the smeared-mass model and approximate in the standard treatment (the narrow-width approximation and convolution method).

The factorization of the cross section (12) and width (17) is a direct consequence of the factorization of the amplitude squared (11). In Refs. [35, 36], the formulae (12) and (17) were obtained in some special cases (for specific tree processes in the case of the lowest-spin UP, $j_R = 0, 1/2, 1$). In Appendices A and B these relations are derived in the most general case—for any spin of UP, any set of initial and final particles, and any structure of the vertices involved.

It should be noted that for scalar and fermion UP ($j_R = 0, 1/2, 3/2, \dots$) the differential cross sections and decay widths also take a factorized form (see Appendix A).

IV. FACTORIZATION METHOD IN THE MODEL OF UP'S WITH SMEARED MASS

The method is based on exact factorization (12), (17) of the simplest processes with UP in an intermediate state that were considered in Sec. III (see also [37, 38]). It can be applied to complicated decay chains and scattering processes that can be reduced to a tree constructed from the basic subdiagrams depicted in Figs. 1 and 2 (loop contributions are limited to the vertex blobs). Further, we consider some examples of such processes with UP's in time-like intermediate states.

FIG. 4. Factorization in $a \rightarrow bcX_F$ decay diagram.

A. Process $ab \rightarrow R_1 \rightarrow cR_2 \rightarrow cX_F$

The cross section of this process (Fig. 3) is a combination of the expressions (12) and (17):

$$\begin{aligned} \sigma(ab \rightarrow cX_F) &= \frac{16L_{R_1}}{L_a L_b \lambda^2(m_a, m_b; \sqrt{s})} \frac{\Gamma_{R_1}^{ab}(s)}{|P_{R_1}(s)|^2} \\ &\times \int_{q_1^2}^{q_2^2} \Gamma(R_1(s) \rightarrow cR_2(q)) \frac{q \Gamma_{R_2}^{X_F}(q)}{|P_{R_2}(q)|^2} dq^2, \end{aligned} \quad (19)$$

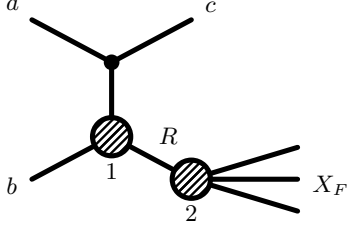
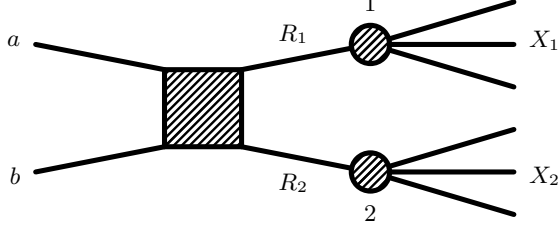
where $L_i = 2j_i + 1$ and $\Gamma_A^B(q) = \Gamma(A(q) \rightarrow B)$. It should be noted that the factorization reduces the number of independent kinematic variables to be integrated over. For example, in the general case of the $2 \rightarrow 3$ process there are $N = 5$ variables that uniquely specify a point in the phase space, four of which have to be integrated over [39]. Some of these variables can be easily integrated out if the process possesses specific symmetry. In the framework of the approach suggested, the number of variables being integrated is always $N' = 1$ (the variable q^2 in Eq. (19)).

B. Decay chain

The width of the decay chain $a \rightarrow bR_1 \rightarrow bcR_2 \rightarrow bcX_F$ depicted in Fig. 4 is a direct consequence of Eq. (17):

$$\begin{aligned} \Gamma(a \rightarrow bcX_F) &= \int_{q_1^2}^{q_2^2} \frac{q \Gamma(a \rightarrow bR_1(q))}{\pi |P_{R_1}(q)|^2} \\ &\times \int_{g_1^2}^{g_2^2} \Gamma(R_1(q) \rightarrow cR_2(g)) \frac{g \Gamma(R_2(g) \rightarrow X_F)}{\pi |P_{R_2}(g)|^2} dg^2 dq^2. \end{aligned} \quad (20)$$

Note that in the general case the number of kinematic variables which uniquely specify a point in the four-particle phase space is $N = 5$ [39], while the model under consideration leaves $N' = 2$ (invariants q^2 and g^2).

FIG. 5. Factorization in $ab \rightarrow cR \rightarrow cX_F$ process.FIG. 6. $R_1 R_2$ -pair production process.

C. Decay of the resonance produced in t -channel scattering

The cross section of the process $ab \rightarrow cR \rightarrow cX_F$ (Fig. 5) is given by a convolution of the cross section $\sigma(ab \rightarrow cR)$ and the width $\Gamma(R \rightarrow X_F)$:

$$\sigma(ab \rightarrow cX_F) = \int_{q_1^2}^{q_2^2} \sigma(ab \rightarrow cR(q)) \frac{q\Gamma(R(q) \rightarrow X_F)}{\pi|P_R(q)|^2} dq^2. \quad (21)$$

This formula can be applied to the description of the processes $e^+e^- \rightarrow \gamma Z \rightarrow \gamma f \bar{f}$ [40] and $eN \rightarrow e\Delta \rightarrow e\pi N$ [38].

D. Resonance-pair production $ab \rightarrow R_1 R_2 \rightarrow X_1 X_2$

Now, we consider the process of boson-pair production depicted in Fig. 6 (e.g. four-fermion production in the double-pole approximation). Applying the model to the process $e^+e^- \rightarrow R_1 R_2$ directly or using the factorization method for the whole process $ab \rightarrow R_1 R_2 \rightarrow X_1 X_2$ leads to the following expression for the inclusive cross section at the tree level [31]:

$$\sigma_{\text{tree}}(e^+e^- \rightarrow R_1 R_2) = \int \int \rho_1(m_1) \rho_2(m_2) \times \sigma_{\text{tree}}[e^+e^- \rightarrow R_1(m_1)R_2(m_2)] dm_1^2 dm_2^2, \quad (22)$$

where $\sigma^{\text{tr}}[e^+e^- \rightarrow R_1(m_1)R_2(m_2)]$ is a cross section for the case of the fixed boson masses m_1 and m_2 , while the probability density function of a mass $\rho(m)$ is defined by the expression

$$\rho_R(m) = \frac{1}{\pi} \frac{m\Gamma_R^{\text{tot}}(m)}{(m^2 - M_R^2)^2 + (m\Gamma_R^{\text{tot}}(m))^2}. \quad (23)$$

These expressions can be applied to the processes $e^+e^- \rightarrow ZZ, W^+W^-$ [41, 42] and $e^+e^- \rightarrow ZH$ [40]. To describe exclusive processes $e^+e^- \rightarrow R_1 R_2 \rightarrow f_i f_{i'} \bar{f}_k \bar{f}_{k'}$, we have to substitute the decay width $\Gamma_R^i = \Gamma(R \rightarrow f_i \bar{f}_{i'})$ for the total width $\Gamma_R^{\text{tot}}(m)$ in the numerator of the right-hand side of Eq. (23) (double-pole approximation).

Using factorization method one can describe complicated decay-chain and scattering processes in a simple way. The same results could be obtained as approximations within the framework of the standard treatment, such as the narrow-width approximation [13, 22], convolution method [23–28], decay-chain method [23], and semi-analytical approach [13]. All these approximations get a strict analytical formulation within the framework of the factorization method. For instance, the narrow-width approximation includes five assumptions, which were considered in detail in [22], while the factorization method contains only one assumption—non-factorable corrections are small (the fifth assumption of the narrow-width approximation [22]).

It is important to estimate an error of calculations in the model with smeared masses that is the deviation of the model results from the standard ones. For a scalar UP the error equals zero. For a vector UP the error comes from the following difference:

$$\eta_{\mu\nu}(q^2) - \eta_{\mu\nu}(m^2) = q_\mu q_\nu \frac{m^2 - q^2}{m^2 q^2}. \quad (24)$$

In the case of the meson-pair production $e^+e^- \rightarrow \rho^0, \omega, \phi \rightarrow \pi^+\pi^-, K^+K^-, \rho^+\rho^-, \dots$ the deviation equals to zero, due to vanishing contribution of the transverse parts of the amplitudes in cases:

$$\begin{aligned} \mathcal{M}^{\text{trans}}(q) &\sim \bar{e}^-(\mathbf{p}_1) \not{q} e^-(\mathbf{p}_2) \\ &= \bar{e}^-(\mathbf{p}_1) (\not{p}_1 + \not{p}_2) e^-(\mathbf{p}_2) = 0. \end{aligned} \quad (25)$$

In the case of the high-energy collisions $e^+e^- \rightarrow Z^0 \rightarrow f \bar{f}$ (we neglect γZ interference) the transverse part of the amplitude is [42]

$$\begin{aligned} \mathcal{M}^{\text{trans}}(q) &\sim \bar{e}^-(\mathbf{p}_1) \not{q} (c_e - \gamma_5 d_e) e^-(\mathbf{p}_2) \\ &\times \bar{f}^+(\mathbf{k}_1) (c_f - \gamma_5 d_f) f^+(\mathbf{k}_2). \end{aligned} \quad (26)$$

We then get at $q^2 \approx M_Z^2$

$$\delta\mathcal{M} \sim \frac{m_e m_f}{M_Z^2} \frac{M_Z - q}{M_Z}. \quad (27)$$

Thus, an error of the factorization method at the vicinity of resonance is always small. The similar estimations can be easily done for the case of a spinor UP.

The relative deviation of the partial cross section for the boson-pair production with consequent decays of the bosons to fermion pairs is

$$\epsilon_f \sim 4 \frac{m_f}{M} \left[1 - M \int_{m_f^2}^s \frac{\rho(q^2)}{q} dq^2 \right], \quad (28)$$

where M is a boson mass. If the fermion f is a τ lepton, the deviation is maximal, $\epsilon_\tau \sim 10^{-3}$. It should be noted that the deviations which are caused by the approach at the tree level are significantly smaller than the errors related to the uncertainties in taking account of radiative corrections [40].

In the case of μ and τ decays the relative deviations of the widths from the standard results are as follows:

$$\begin{aligned} \epsilon(\mu \rightarrow e\nu\bar{\nu}) &\approx 5 \cdot 10^{-4}, & \epsilon(\tau \rightarrow e\nu\bar{\nu}) &\approx 3 \cdot 10^{-6}, \\ \epsilon(\tau \rightarrow \mu\nu\bar{\nu}) &\approx 3 \cdot 10^{-2}. \end{aligned} \quad (29)$$

The error in the last case is noticeable because of the factor m_μ/m_τ . In the case of a spinor UP the deviation is of the order of $(M_f - q)/M_f$. It could be substantial if q is well far from the resonance region, but this value is suppressed by averaging over variable mass as in (28).

V. CONCLUSION

The factorization method is a convenient analytical way to describe decays and scattering processes. The factorization simplifies calculations considerably and gives compact universal formulae for decay widths and cross sections of complicated processes.

In this work, we have shown that in the model with smeared masses the exact factorization holds for a time-like intermediate states of arbitrarily high spin. The factorization is due to the specific tensor-spinor structure of the propagators in the model. We derived the factorization formulae for some particular cascade processes. The model of UP with smeared mass is an analytical analog of the narrow-width approximation. This phenomenological approach can be considered as a convenient approximation to the conventional field-theory calculations which is always valid in the vicinity of the resonance peak, if non-resonance contribution are negligible. Note also that the method allows us to take account of all factorable higher-order corrections.

Appendix A: Factorization of amplitude squared

The normalization of the amplitudes can be defined by their relations to the observables. Differential width of the decay process $a(p) \rightarrow X_F(k_1, \dots, k_f)$, where X_F is a set of f final particles with 4-momenta k_i , $i = 1, \dots, f$, has the form:

$$\begin{aligned} d\Gamma(p, k_1, \dots, k_f) \\ = \frac{1}{2\pi K^2} \delta\left(p - \sum k_i\right) |\mathcal{M}(p, k_1, \dots, k_f)|^2 \prod d\mathbf{k}_i. \end{aligned} \quad (A1)$$

where $K = (2\pi)^{3(s+f)/2+4(m-n)}$. Here, $s = 1$ and f are numbers of initial and final states, m and n are numbers of internal lines and vertices.

Differential cross section of the process $a(p_a)b(p_b) \rightarrow X_F(k_1, \dots, k_f)$ is:

$$\begin{aligned} d\sigma(p_a, p_b, k_1, \dots, k_f) \\ = \frac{(2\pi)^2}{v(p)K^2} \delta\left(q - \sum k_i\right) |\mathcal{M}(p_a, p_b, k_1, \dots, k_f)|^2 \prod d\mathbf{k}_i, \end{aligned} \quad (A2)$$

where $q = p_a + p_b$ and $v(p)p_a^0 p_b^0 = \sqrt{(p_a p_b)^2 - m_a^2 m_b^2}$. The amplitude $\mathcal{M}(p, k)$ does not contain any (2π) and is normalized by $(2p_0)^{-1/2}$ for the boson states. For the spinor states, the same normalizing coefficients in the amplitude squared come from the polarization matrices (3) and (4).

1. UP with a spin $j_R = 1/2$

Let us consider a process $X_I \rightarrow R \rightarrow X_F$, where R is a spin-1/2 UP with a large width. To reduce the number of nonessential indices in the following equations, we restrict ourselves to the case of a two-particle initial X_I and multi-particle final state X_F consisting of one spin-1/2 particle and a number of scalar particles (see Fig. 1). As it will be easily seen, however, all the following results are valid for X_I and X_F including any number of particles of arbitrarily high spins. The vertices (1) and (2) in Fig. 1 are not specified and can involve loops. The amplitude of this process is written as follows:

$$\begin{aligned} \mathcal{M}(p, q, k) &= \frac{1}{\sqrt{2p_b^0}} \bar{\psi}_F^+(\mathbf{k}_1) \Gamma_{(2)}(k, q) \frac{\not{q} + q}{P_R(q)} \\ &\times \Gamma_{(1)}(p, q) \psi_I^-(\mathbf{p}_a) \prod_{i=2}^f \frac{1}{\sqrt{2k_i^0}}. \end{aligned} \quad (A3)$$

Here, $p = p_a, p_b$ and $k = k_1, k_2, \dots, k_f$ are momenta of the initial and final particles, respectively. Hermitian conjugation of the amplitude $\mathcal{M}^\dagger(p, q, k)$ is taken so that equalities $(\bar{\psi}_1^+ \Gamma \psi_2^-)^\dagger = \bar{\psi}_2^+ \bar{\Gamma} \psi_1^-$ and $\bar{\Gamma} = \gamma_0 \Gamma^+ \gamma_0$ hold. The amplitude squared $|\mathcal{M}(p, q, k)|^2$ (summed over the spin states of the final particles and averaged over the spin states of the initial ones) is as follows:

$$\begin{aligned} |\mathcal{M}(p, q, k)|^2 &= \frac{(2p_a^0 2p_b^0 \prod 2k_i^0)^{-1}}{(2j_a + 1)(2j_b + 1)} \frac{1}{|P_R(q)|^2} \\ &\times \text{Tr} \left[(\not{p}_a + m_a) \bar{\Gamma}_{(1)} (\not{q} + q) \bar{\Gamma}_{(2)} \right. \\ &\quad \left. \times (\not{k}_1 + m'_1) \Gamma_{(2)} (\not{q} + q) \Gamma_{(1)} \right]. \end{aligned} \quad (A4)$$

In Eq. (A4), the spins of the initial particles $j_a = 1/2$ and $j_b = 0$ are preserved as symbols for generality.

We can also write the amplitudes of the decays $\psi_R \rightarrow$

X_I and $\psi_R \rightarrow X_F$

$$\begin{aligned}\mathcal{M}_1(p, q) &\equiv \mathcal{M}(\psi_R(q) \rightarrow X_I) \\ &= \frac{1}{\sqrt{2p_b^0}} \bar{\psi}_I^+(\mathbf{p}_a) \bar{\Gamma}_{(1)}(p, q) \psi_R^-(\mathbf{q}), \\ \mathcal{M}_2(k, q) &\equiv \mathcal{M}(\psi_R(q) \rightarrow X_F) \\ &= \prod_{i=2}^f \frac{1}{\sqrt{2k_i^0}} \cdot \bar{\psi}_F^+(\mathbf{k}_1) \Gamma_{(2)}(k, q) \psi_R^-(\mathbf{q}).\end{aligned}\quad (\text{A5})$$

The amplitudes squared $|\mathcal{M}_{1,2}|^2$ are as follows:

$$\begin{aligned}|\mathcal{M}_1(p, q)|^2 &\equiv \text{Tr} \hat{M}_1(p, q) = \frac{1}{8L_R p_a^0 p_b^0 q^0} \\ &\quad \times \text{Tr}[(\not{q} + q) \Gamma_{(1)}(\not{p}_a + m_a) \bar{\Gamma}_{(1)}]; \\ |\mathcal{M}_2(k, q)|^2 &\equiv \text{Tr} \hat{M}_2(k, q) = \frac{1}{L_R (2q^0) \prod (2k_i^0)} \\ &\quad \times \text{Tr}[(\not{q} + q) \bar{\Gamma}_{(2)}(\not{k}_1 + m'_1) \Gamma_{(2)}],\end{aligned}\quad (\text{A6})$$

where $L_R = 2J_R + 1$. The above equations define 4×4 matrices $\hat{M}_1(p, q)$ and $\hat{M}_2(k, q)$ that have a structure $\hat{M}_{1,2} = \hat{M}'_{1,2}(\not{q} + q)$.

Now we can prove the following identity:

$$\begin{aligned}J[|\mathcal{M}(p, q, k)|^2] &= \frac{L_R^2 (2q^0)^2}{L_a L_b |P_R(q)|^2} \text{Tr} J[\hat{M}_2(k, q) \hat{M}_1(p, q)] \\ &= \frac{L_R (2q^0)^2}{L_a L_b |P_R(q)|^2} \text{Tr} \hat{M}_1(k, q) \cdot J[\text{Tr} \hat{M}_2(p, q)].\end{aligned}\quad (\text{A7})$$

Here, $J[X]$ stands for the integration over the phase space of the final state X_F

$$J[X] \equiv \int X \delta\left(q - \sum k_i\right) \prod d\mathbf{k}_i. \quad (\text{A8})$$

To this end, let us consider the trace in the right-hand side of Eq. (A7), $T(p, q) \equiv \text{Tr} J[\hat{M}_2(k, q) \hat{M}_1(p, q)]$. Since $(\not{q} + q)^2 = 2q(\not{q} + q)$, we get that

$$\begin{aligned}T(p, q) &= \frac{1}{4q^2} \text{Tr} J[\hat{M}_2(k, q)(\not{q} + q) \hat{M}_1(p, q)(\not{q} + q)] \\ &\equiv \frac{1}{4q^2} \text{Tr}[\hat{A}(q) \hat{M}_1(p, q)],\end{aligned}\quad (\text{A9})$$

where $\hat{A}(q) = (\not{q} + q) J[\hat{M}_2(k, q)](\not{q} + q)$ and the following identity holds:

$$\hat{A}(q) = \frac{1}{4q^2} (\not{q} + q) \hat{A}(q) (\not{q} + q). \quad (\text{A10})$$

The matrix $\hat{A}(q)$ can be expanded in terms of linearly independent matrices as follows:

$$\hat{A}(q) = (\not{q} + q)(a_1 + b_1 \gamma_5) + (\not{q} - q)(a_2 + b_2 \gamma_5). \quad (\text{A11})$$

Here $a_i = a_i(q)$ and $b_i = b_i(q)$ are scalar coefficients. Substituting the decomposition (A11) into the righthand

side of Eq. (A10), we get that $\hat{A}(q) = a_1(\not{q} + q)$. As a result, we have

$$\hat{A}(q) = q(\not{q} + q) \text{Tr} J[\hat{M}_2(k, q)] \quad (\text{A12})$$

and consequently

$$T(p, q) = \frac{1}{2} \text{Tr} J[\hat{M}_2(k, q)] \cdot \text{Tr} \hat{M}_1(p, q), \quad (\text{A13})$$

where we have used the structure $\hat{M}_1(p, q) = \hat{M}'_1(p, q)(\not{q} + q)$. Finally, we have proved the identity (A7) and the factorization of the amplitude squared for the case of a spin-1/2 UP ($2j_R + 1 = 2$):

$$\begin{aligned}J[|\mathcal{M}(p, q, k)|^2] &= \frac{L_R (2q_0)^2}{L_a L_b |P_R(q)|^2} \\ &\quad \times |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2].\end{aligned}\quad (\text{A14})$$

2. UP with a spin $j_R = 1$

Now, we proceed to a process $X_I \rightarrow V_R \rightarrow X_F$ with a vector UP in the intermediate state. In the case of scalar initial and final particles, the amplitude is

$$\begin{aligned}\mathcal{M}(p, q, k) &= \frac{1}{\sqrt{2p_a^0}} \frac{1}{\sqrt{2p_b^0}} \prod_{i=1}^f \frac{1}{\sqrt{2k_i^0}} \\ &\quad \times \Gamma_\mu^{(1)}(p, q) \frac{\eta^{\mu\nu}(q)}{P_R(q)} \Gamma_\nu^{(2)}(k, q).\end{aligned}\quad (\text{A15})$$

Here, $\Gamma_\mu^{(1,2)}$ are vertexes; $\eta_{\mu\nu}(q) = -g_{\mu\nu} + q_\mu q_\nu / q^2$. For the amplitude squared we then have

$$\begin{aligned}|\mathcal{M}(p, q, k)|^2 &= \frac{[4p_a^0 p_b^0 \prod (2k_i^0)]^{-1}}{(2j_a + 1)(2j_b + 1) |P_R(q)|^2} \frac{1}{|P_R(q)|^2} \\ &\quad \times \Gamma_{\mu'}^{(1)*}(p, q) \Gamma_\mu^{(1)}(p, q) \eta^{\mu\nu}(q) \\ &\quad \times \eta^{\mu'\nu'}(q) \Gamma_{\nu'}^{(2)*}(q, k) \Gamma_\nu^{(2)}(q, k).\end{aligned}\quad (\text{A16})$$

For the processes $V_R \rightarrow X_I$ and $V_R \rightarrow X_F$ the amplitudes and their squares can be written as

$$\mathcal{M}_1(p, q) = \frac{1}{\sqrt{2q^0}} \frac{1}{\sqrt{2p_a^0}} \frac{1}{\sqrt{2p_b^0}} e^\mu(\mathbf{q}) \Gamma_\mu^{(1)}(p, q), \quad (\text{A17})$$

$$\mathcal{M}_2(k, q) = \frac{1}{\sqrt{2q^0}} \prod_{i=1}^f \frac{1}{\sqrt{2k_i^0}} \cdot e^\nu(\mathbf{q}) \Gamma_\nu^{(2)}(k, q) \quad (\text{A18})$$

and

$$|\mathcal{M}_1(p, q)|^2 = \frac{(8q^0 p_a^0 p_b^0)^{-1}}{2j_R + 1} \Gamma_{\mu'}^{(1)*}(p, q) \Gamma_\mu^{(1)}(p, q) \eta^{\mu\mu'}(q), \quad (\text{A19})$$

$$|\mathcal{M}_2(k, q)|^2 = \frac{[2q^0 \prod (2k_i^0)]^{-1}}{2j_R + 1} \Gamma_{\nu'}^{(2)*}(q, k) \Gamma_\nu^{(2)}(q, k) \eta^{\nu\nu'}(q). \quad (\text{A20})$$

For brevity we introduce tensors $T_{\mu\mu'}^{(1)}(p, q)$ and $T_{\nu\nu'}^{(2)}(k, q)$ so that Eqs. (A16), (A19), and (A20) take the form

$$|\mathcal{M}(p, k, q)|^2 = \frac{(2j_R + 1)^2 (2q^0)^2}{(2j_a + 1)(2j_b + 1)|P_R(q)|^2} \times T_{\mu\mu'}^{(1)}(p, q) \eta^{\mu\nu}(q) \eta^{\mu'\nu'}(q) T_{\nu\nu'}^{(2)}(k, q), \quad (\text{A21})$$

$$|\mathcal{M}_1(p, q)|^2 = T_{\mu\mu'}^{(1)}(p, q) \eta^{\mu\mu'}(q), \quad (\text{A22})$$

$$|\mathcal{M}_2(k, q)|^2 = T_{\nu\nu'}^{(2)}(k, q) \eta^{\nu\nu'}(q). \quad (\text{A23})$$

We can show that the exact factorization arises as a result of integrating over the phase space of the final state X_F

$$J[|\mathcal{M}(p, k, q)|^2] = \frac{(2j_R + 1)^2 (2q^0)^2}{(2j_a + 1)(2j_b + 1)|P_R(q)|^2} \times T_{\mu\mu'}^{(1)}(p, q) \eta^{\mu\nu}(q) \eta^{\mu'\nu'}(q) J[T_{\nu\nu'}^{(2)}(k, q)]. \quad (\text{A24})$$

The tensor integral $J[T_{\nu\nu'}^{(2)}(k, q)]$ can be decomposed in the following way:

$$J[T_{\nu\nu'}^{(2)}(k, q)] = \frac{1}{2j_R + 1} [T(q) \eta_{\nu\nu'}(q) + S(q) q_\nu q_{\nu'}]. \quad (\text{A25})$$

The second term does not contribute to Eq. (A24) because of the property $\eta^{\mu\nu}(q) q_\nu = 0$. Multiplying Eq. (A25) by $\eta^{\nu\nu'}(q)$, we find

$$T(q) = J[T_{\nu\nu'}^{(2)}(k, q) \eta^{\nu\nu'}(q)] = J[|\mathcal{M}_2(k, q)|^2]. \quad (\text{A26})$$

Finally, substituting Eqs. (A25) and (A26) into (A24) gives

$$J[|\mathcal{M}(p, k, q)|^2] = \frac{(2j_R + 1)(2q^0)^2}{(2j_a + 1)(2j_b + 1)|P_R(q)|^2} \times |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2]. \quad (\text{A27})$$

Therefore, we have proved that being integrated over the final momenta the transition probability of a process with a vector UP in the intermediate time-like state is factorized exactly in the framework of the model with the propagator (1).

3. UP with a spin $j_R \geq 3/2$

We should also discuss the case of UP carrying arbitrarily high spin j_R . In the model of UP with smeared mass, the tensor-spinor structure of the propagators of higher-spin particles is given by the projection operators introduced in Refs. [43, 44]. For integer spins $j_R = \ell = 1, 2, \dots$, the simplest projector $\ell = 1$ is well known:

$$P_{\mu\nu}^{(1)}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} = -\eta_{\mu\nu}(q). \quad (\text{A28})$$

The projection operators for higher integer spins $\ell \geq 2$ are defined by the recurrence relation

$$P_{\bar{\mu}\bar{\nu}}^{(\ell)}(q) = \frac{1}{\ell^2} \left\{ P_{\bar{\mu}^f \bar{\nu}^g}^{(\ell-1)}(q) P_{\mu_f \nu_g}^{(1)}(q) - \frac{1}{2(2\ell-1)} \left[P_{\bar{\mu}^f \mu_f \bar{\nu}^g h}^{(\ell-1)}(q) P_{\nu_{gh}}^{(1)}(q) + P_{\bar{\nu}^f \nu_f \bar{\mu}^g h}^{(\ell-1)}(q) P_{\mu_{gh}}^{(1)}(q) \right] \right\}, \quad (\text{A29})$$

where $\bar{\mu} = \mu_1 \mu_2 \dots \mu_\ell$, $\bar{\mu}^f = \mu_1 \mu_2 \dots \mu_{f-1} \mu_{f+1} \dots \mu_\ell$, $\mu_{fg} = \mu_f \mu_g$ are multi-indices. The summation over repeated Latin indices f, g, h of the Greek multi-indices is implied from 1 to ℓ .

The projectors possess the following properties [43, 44]

$$P_{\mu_1 \dots \mu_a \mu_b \dots \mu_\ell \bar{\nu}}^{(\ell)} = P_{\mu_1 \dots \mu_b \mu_a \dots \mu_\ell \bar{\nu}}, \quad P_{\bar{\mu}\bar{\nu}}^{(\ell)} = P_{\bar{\nu}\bar{\mu}}^{(\ell)}, \\ P_{\bar{\mu}\bar{\lambda}}^{(\ell)} P_{\bar{\nu}}^{(\ell)} = P_{\bar{\mu}\bar{\nu}}^{(\ell)}, \quad P_{\bar{\lambda}}^{(\ell)} = 2\ell + 1, \quad (\text{A30}) \\ q^{\mu_a} P_{\bar{\mu}\bar{\nu}}^{(\ell)} = 0, \quad g^{\mu_a \mu_b} P_{\bar{\mu}\bar{\nu}}^{(\ell)} = 0$$

for any $a, b = 1, \dots, \ell$. For $\ell = 1, 2, 3, 4$ the relations (A30) are checked straightforwardly. For $\ell \geq 4$ these properties can be proved by induction.

The projectors for half-integer spins $j_R = \ell + 1/2 = 3/2, 5/2, \dots$ are written as

$$P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) = \frac{\ell+1}{2\ell+3} \gamma^\alpha \gamma^\beta P_{\bar{\mu}\alpha\bar{\nu}\beta}^{(\ell+1)}(q), \quad \ell \geq 1. \quad (\text{A31})$$

For $\ell = 1$ and $\ell = 2$ the projectors (A31) coincide with the known expressions for $P_{\mu\nu}^{(3/2)}$ [45] and $P_{\mu_1 \mu_2 \nu_1 \nu_2}^{(5/2)}$ [46].

The properties of the projectors (A31) are

$$P_{\bar{\mu}\bar{\lambda}}^{(\ell+1/2)} P_{(\ell+1/2)\bar{\nu}}^{(\ell+1/2)} = P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}, \quad \bar{P}_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) = \bar{P}_{\bar{\nu}\bar{\mu}}^{(\ell+1/2)}(q), \\ \text{Tr } P_{\bar{\mu}}^{(\ell+1/2)\bar{\mu}} = 4(\ell+1), \quad \not{q} P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) = P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) \not{q}, \\ q^{\mu_a} P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) = 0, \quad \gamma^{\mu_a} P_{\bar{\mu}\bar{\nu}}^{(\ell+1/2)}(q) = 0. \quad (\text{A32})$$

These identities follow from the identities (A30) for the projectors $P_{\bar{\mu}\bar{\nu}}^{(\ell)}$. It is the properties (A30) and (A32) which allow us to generalize the consideration of the previous subsections to the case of arbitrary spins.

Now, it is easy to see that the factorized formula (A27) for the amplitude squared is valid for any integer spin of UP. Indeed, in the proof of Appendix A2 we just need to change all Lorentz indices to multi-indices ($\mu \rightarrow \bar{\mu}$, $\nu \rightarrow \bar{\nu}$, etc.) and spin-1 projectors to spin- ℓ ones ($\eta_{\mu\nu} \rightarrow -P_{\bar{\mu}\bar{\nu}}^{(\ell)}$). By similar substitutions we can generalize the proof of the relation (A27) to arbitrarily high half-integer spins.

Appendix B: Factorized formula for cross section and convolution formula for decay width

In this Appendix we derive universal factorized formula for the cross section of the scattering $ab \rightarrow R \rightarrow X_F$ and convolution formula for the decay $a \rightarrow bR \rightarrow bX_F$.

Using Eqs. (A1), (A2), and (A27), we find the cross section of the process under consideration

$$\begin{aligned}\sigma(p_1, p_2) &= \frac{(2\pi)^2}{K^2 v(p)} J[|\mathcal{M}(p, k)|^2] \\ &= \frac{(2j_R + 1)(2q^0)^2 |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2]}{(2\pi) K_2^2 (2j_a + 1)(2j_b + 1) v(p) |P_R(q)|^2} \\ &= \frac{(2j_R + 1)(2q^0)^2 |\mathcal{M}_1(p, q)|^2 \Gamma(R(q) \rightarrow X_F)}{(2j_a + 1)(2j_b + 1) v(p) |P_R(q)|^2},\end{aligned}\quad (\text{B1})$$

where $K = (2\pi)^{3f/2-1} = 2\pi K_1 K_2$ and $K_1 = (2\pi)^{1/2}$. The amplitude squared $|\mathcal{M}_1(p, q)|^2$ is related to the decay width

$$\begin{aligned}\Gamma(R(q) \rightarrow ab) &= \frac{1}{(2\pi)^2} \int \delta(q^0 - p_a^0 - p_b^0) |\mathcal{M}_1(p, q)|^2 d\mathbf{p}_a \\ &= \frac{1}{\pi} |\mathcal{M}^{(1)}(p, q)|^2 p_a^0 p_b^0 \frac{p}{q^0}.\end{aligned}\quad (\text{B2})$$

In the center-of-mass frame, it follows from (B2) that

$$|\mathcal{M}_1(p, q)|^2 = 2\pi \frac{\Gamma(R(q) \rightarrow a_1 a_2)}{p_a^0 p_b^0 \lambda(m_1^2, m_2^2; q^2)}, \quad (\text{B3})$$

where the function $\bar{\lambda}(m_a^2, m_b^2; q^2) = 2v(p) p_a^0 p_b^0 / q^2$ is defined by Eq. (13).

Upon substituting Eq. (B3) into (B1) we get

$$\begin{aligned}\sigma(q^2) &= \frac{16\pi(2j_R + 1)}{(2j_a + 1)(2j_b + 1) \bar{\lambda}^2(m_a^2, m_b^2; q^2) |P_R(q^2)|^2} \\ &\quad \times \Gamma(R(q) \rightarrow ab) \Gamma(R(q) \rightarrow X_F).\end{aligned}\quad (\text{B4})$$

Universal factorized formula (B4) describes cross section of the process $ab \rightarrow R \rightarrow X_F$, where R is UP with arbitrarily high spin in the time-like intermediate state, X_F is any set of final particles, and the vertices $\Gamma_{(1,2)}$ have an arbitrary (loop) structure. This result is a generalization of the corresponding formula for the tree processes

$ab \rightarrow R \rightarrow cd$ that was obtained for some particular interactions mediated by scalar, vector, and spinor UP R in Ref. [36].

Now, let us consider the decay $a \rightarrow bR \rightarrow bX_F$ with all particles and vertices being of any type. The factorized relation for the amplitude squared of the decay differs from that for the scattering amplitude (A27) by a numeric factor

$$\begin{aligned}J[|\mathcal{M}(p, k, q)|^2] &= \frac{(2q^0)^2}{|P_R(q)|^2} |\mathcal{M}_1(p, q)|^2 J[|\mathcal{M}_2(k, q)|^2],\end{aligned}\quad (\text{B5})$$

In Eq. (B5) the amplitudes $\mathcal{M}_1(p, q)$ and $\mathcal{M}_2(k, q)$ correspond to the subprocesses $a \rightarrow bR$ and $R \rightarrow X_F$, respectively. In terms of the normalized amplitudes $\mathcal{A}_i = \mathcal{M}_i / K_i$, the decay width (A1) is

$$\begin{aligned}\Gamma &= \frac{1}{(2\pi)^3} \int \frac{(2q^0)^2}{|P_R(q)|^2} |\mathcal{A}_1(p, q)|^2 |\mathcal{A}_2(k, q)|^2 \\ &\quad \times \delta(p - k_1 - q) d\mathbf{k}_1 d\mathbf{k}_2 \cdots d\mathbf{k}_f \\ &= \frac{4}{(2\pi)^2} \int \frac{(q^0)^2}{|P_R(q)|^2} |\mathcal{A}_1(p_1, q)|^2 \Gamma(R(q) \rightarrow X_F) d\mathbf{k}_1,\end{aligned}\quad (\text{B6})$$

where $q = k_2 + k_3 + \cdots + k_f$. Then, we find from Eq. (B3) that

$$\Gamma(a \rightarrow bR(q)) = 2|\mathcal{A}_1(p, q)|^2 |\mathbf{k}_1| k_1^0 \frac{q^0}{p^0}, \quad (\text{B7})$$

where $\mathcal{A}_1 = \mathcal{M}_1 / \sqrt{2\pi}$. Substituting (B7) into (B6) and noting that $d\mathbf{k}_1 = 2\pi dq^2 |\mathbf{k}_1| k_1^0 / p^0$ in the rest frame of the decayed particle a , we get

$$\begin{aligned}\Gamma(a \rightarrow bR \rightarrow bX_F) &= \int \Gamma(a \rightarrow bR(q)) \frac{q\Gamma(R(q) \rightarrow X_F)}{\pi |P_R(q)|^2} dq^2.\end{aligned}\quad (\text{B8})$$

Here, the limits of integration are determined by the kinematics of the process and we have used the relation $q\Gamma(q) = q^0 \Gamma(q^0)$, where $\Gamma(q^0)$ is the width in the center-of-mass frame, $\mathbf{q} = 0$. Summation over all decay channels of UP $R(q)$ gives:

$$\Gamma(a \rightarrow bR) = \int \Gamma(a \rightarrow bR(q)) \rho_R(q) dq^2, \quad (\text{B9})$$

where $\rho_R(q) = q\Gamma^{\text{tot}} R(q) / (\pi |P_R(q)|^2)$ is interpreted in the framework of the model of UP with smeared mass as the probability density of the mass parameter $m^2 = q^2$.

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